Lecture 39

$\boxed{\text{Ex}}$

$$y = x^2 - 4$$
$$x = y^2 - 5$$

- Can use "linear approximation" to address this system.

- <u>Guess</u> : $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

  How to improve this ?

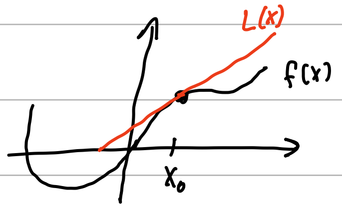
$\boxed{\text{Pre-Ex}}$

$y = f(x) = x^2$

$x_0 = 2$

$x = x_0 + \Delta x$

$$\boxed{f(x) \approx f(x_0) + f'(x_0)\Delta x = L(x)}$$



$f(x) = x^2 = 5$

Guess $x_0 = 2$

$5 = x^2 \approx f(x_0) + f'(x_0)\Delta x = 4 + 4\Delta x$

$x_1 := x = x_0 + \Delta x = 2 + \frac{1}{4} = \frac{9}{4}$

Linear Approximation for fns $f: \mathbb{R}^n \to \mathbb{R}^m$

Case 1: $f: \mathbb{R} \to \mathbb{R}^m$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \approx \begin{pmatrix} f_1(x_0) \\ \vdots \\ f_m(x_0) \end{pmatrix} + \begin{pmatrix} f_1'(x_0) \\ \vdots \\ f_m'(x_0) \end{pmatrix} \Delta x$$

$$= \overrightarrow{f(x_0)} + \overrightarrow{f'(x_0)}\, \Delta x$$

Want to solve:

$$\overrightarrow{F\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} x^2 - y \\ x^2 - x \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$$

- Now given $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, write $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

Linear Approximation

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} \approx \vec{F}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \frac{\partial \vec{F}}{dx}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Delta x + \frac{\partial \vec{F}}{dy}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Delta y$$

In Example:

$$\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} \approx \vec{F}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} 2x_0 \\ -1 \end{pmatrix} \Delta x + \begin{pmatrix} -1 \\ 2y_0 \end{pmatrix} \Delta y$$

$$\approx \vec{F}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{bmatrix} 2x_0 & -1 \\ -1 & 2y_0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

· Try $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and Solve

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} \approx \vec{F}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 6 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0.695 \\ -0.217 \end{pmatrix}$$

Get new guess at solution:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2.695 \\ 2.783 \end{pmatrix}$$

So now:

$$\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{F}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{bmatrix} 2x_1 & -1 \\ -1 & 3y_1 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -0.0944858 \\ -0.0254698 \end{pmatrix}$$

$$\text{So } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.60117 \\ 2.75714 \end{pmatrix}$$